Complex Numbers Summary

What does a complex number mean?

A complex number has a ‘real’ part and an ‘imaginary’ part (the imaginary part involves the square root of a negative number).

We use Z to denote a complex number:

\[ Z = x + iy \]

E.g. \( Z = 4 + 3i \)

\( \text{Re}(Z) = 4 \quad \text{Im}(Z) = 3 \)

Example:

Powers of \( i \)

\( i \) stands for \( \sqrt{-1} \) so:

\[ i^2 = (\sqrt{-1})^2 = -1 \]
\[ i^4 = (i^2)^2 = (-1)^2 = 1 \]

For any power of \( i \) take out as many \( i^4 \)'s and \( i^2 \)'s as possible and they will all end up as \( \pm i \) or \( \pm 1 \).

Example:

\[ i^{11} = (i^4)^2 i i = 1^2 \times -1 \times i = -i \]

OR: just take out \( i^2 \)'s if you find it easier to remember.

Example:

\[ i^{33} = (i^2)^{16} i = (-1)^{16} i = i \]

Adding & Subtracting

This is easy – just add or subtract the real part and add or subtract the imaginary parts:

Examples:

\( (4 + 3i) + (2 + 6i) = (6 + 9i) \)
\( (3 + 7i) - (1 - 3i) = (2 + 10i) \)

Multiplying

Multiply out the 2 brackets.

Example:

\( (3 + 5i)(4 - 2i) = 12 - 6i + 20i - 10i^2 = 12 + 14i - 10 \times (-1) = 22 + 14i \)
**Complex Conjugate**

The conjugate is exactly the same as the complex number but with the opposite sign in the middle. When multiplied together they always produce a real number because the middle terms disappear (like the difference of 2 squares with quadratics).

**Example:** \((4 + 6i)(4 - 6i) = 16 - 24i + 24i - 36i^2 = 16 - 36(-1) = 16 + 36 = 52\)

**Dividing**

Dividing by a real number: divide the real part and divide the imaginary part.

Dividing by a complex number: Multiply top and bottom of the fraction by the complex conjugate of the denominator so that it becomes real, then do as above.

**Examples:**
\[
\frac{3+4i}{2} = \frac{3}{2} + \frac{4}{2}i = 1.5 + 2i
\]
\[
\frac{4-5i}{3+2i} = \frac{4-5i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{12-8i-15i+10i^2}{9-6i+6i-4i^2} = \frac{12-23i+10(-1)}{9-4(-1)} = \frac{2-23i}{13} = \frac{2}{13} - \frac{23}{13}i
\]

**Graphical Representation**

A complex number can be represented on an Argand diagram by plotting the real part on the x-axis and the imaginary part on the y-axis.

**Example:**

![Graphical Representation of Complex Numbers](image)

- **Modulus:** is written as \(|z|\) and is the length of OP, therefore \(|z| = \sqrt{x^2 + y^2}\)
- **Argument:** is the angle \(\theta\) that is made with the horizontal axis (denoted by \(\angle\)).

**Polar & Exponential Form**

As well as the basic form \((z = x + iy)\) there are 2 more ways of writing a complex number:

<table>
<thead>
<tr>
<th>Polar: (z = r(cos\theta + isin\theta))</th>
<th>Exponential: (z = re^{i\theta})</th>
</tr>
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Where \(r\) is the length of the line and \(\theta\) is the angle it makes with the x-axis (this should be in radians for the exponential form).

Remember:
- To find the modulus (length), \(r\): use Pythagoras
- To find the argument (angle), \(\theta\): use \(tan^{-1}\left(\frac{y}{x}\right)\)
Converting between the different forms:

<table>
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<tr>
<th>Basic</th>
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<th>Need to find $r$ and $\theta$</th>
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<td></td>
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**Example:** Express $z = 3 + 4i$ in polar and exponential form

Modulus: $r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Argument: $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$ (in radians $\theta = 0.927\pi$)

Polar form: $z = 5(\cos(53.1) + i\sin(53.1))$

Exp form: $z = 5e^{0.927i}$

Nb always do a quick sketch of the complex number and if it’s in a different quadrant adjust the angle as necessary.

**Example:** Express $z = 7e^{\frac{i\pi}{3}}$ in basic form

$x = r\cos\theta \quad \therefore x = 7\cos\left(\frac{\pi}{3}\right) = 3.5$

$y = r\sin\theta \quad \therefore y = 7\sin\left(\frac{\pi}{3}\right) = 6.1$

Basic form: $z = 3.5 + 6.1i$

A reminder of the 3 forms:

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Conversions: $x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
Multiplying with Polar or Exponential form

Let \( z_1 = z_2 \cdot z_3 \)

Then \(|z_1| = |z_2| \times |z_3|\)
And \(\angle z_1 = \angle z_2 + \angle z_3\)

Example: If \( z_1 = 5e^{\frac{n_1}{2}i} \) and \( z_2 = 3e^{\frac{n_2}{3}i} \) find \( z_1z_2 \)

New modulus: \( 5 \times 3 = 15 \)
New angle: \( \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6} \)
\[ \therefore z_1z_2 = 15e^{\frac{5n_1}{6}i} \]

Dividing with Polar or Exponential form

Let \( z_1 = \frac{z_2}{z_3} \)

Then \(|z_1| = \frac{|z_2|}{|z_3|}\)
And \(\angle z_1 = \angle z_2 - \angle z_3\)

Example: If \( z_1 = 5\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right) \) and \( z_2 = 3\left(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right) \) find \( \frac{z_1}{z_2} \)

New modulus: \( 5 \div 3 = \frac{5}{3} \)
New angle: \( \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \)
\[ \therefore \frac{z_1}{z_2} = \frac{5}{3}\left(\cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right) \]

De Moivres Theorem:
Is used for raising a complex number to a power.

\[ z^n = r^n(\cos(n\theta) + isin(n\theta)) \]

e.g If \( z = 3\left(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right) \)
then \( z^5 = 3^5\left(\cos\frac{5\pi}{3} + isin\frac{5\pi}{3}\right) \)

We could use De Moivres or:
\[ (1 + i)^{100} = ((1 + i)^2)^{50} \]

The same method can be used for a root (e.g. \( z^n \)). However, there will be \( n \) answers, all with the same modulus but with different arguments. To find the arguments you need to keep adding \( \frac{2\pi}{n} \) to your previous answer.